

SYMMETRIC AUSLANDER AND BASS CATEGORIES

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ABSTRACT. We define the symmetric Auslander category $A^s(R)$ to consist of complexes of projective modules whose left- and right-tails are equal to the left- and right-tails of totally acyclic complexes of projective modules.

The symmetric Auslander category contains $A(R)$, the ordinary Auslander category. It is well known that $A(R)$ is intimately related to Gorenstein projective modules, and our main result is that $A^s(R)$ is similarly related to what can reasonably be called Gorenstein projective homomorphisms. Namely, there is an equivalence of triangulated categories

$$\underline{\mathbf{GMor}}(R) \xrightarrow{\sim} A^s(R)/K^b(\mathbf{Prj} R)$$

where $\underline{\mathbf{GMor}}(R)$ is the stable category of Gorenstein projective objects in the abelian category $\mathbf{Mor}(R)$ of homomorphisms of R -modules.

This result is set in the wider context of a theory for $A^s(R)$ and $B^s(R)$, the symmetric Bass category which is defined dually.

0. INTRODUCTION

Let R be a commutative noetherian ring with a dualizing complex D . Such complexes were introduced in [5, chp. V] where it was also shown that the functor $\mathbf{RHom}_R(-, D)$ is a contravariant autoequivalence of $D^f(R)$, the finite derived category of R .

Some time later, it was shown in [2, sec. 3] that by restricting to certain subcategories $A(R)$ and $B(R)$ of the derived category $D(R)$, the functors $D \overset{L}{\otimes}_R -$ and $\mathbf{RHom}_R(D, -)$ become quasi-inverse covariant

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equivalences

$$\mathbf{A}(R) \begin{array}{c} \xrightarrow{D^{\mathbf{L}}_{\otimes R} -} \\ \xleftarrow{\mathrm{RHom}_R(D, -)} \end{array} \mathbf{B}(R).$$

The categories $\mathbf{A}(R)$ and $\mathbf{B}(R)$ are known as the Auslander and Bass categories of R . The precise definition is given in Remark 1.5 below, but note that $\mathbf{A}(R)$ and $\mathbf{B}(R)$ contain the bounded complexes of projective, respectively injective, modules.

This paper introduces the symmetric Auslander category $\mathbf{A}^s(R)$ and the symmetric Bass category $\mathbf{B}^s(R)$ which contain $\mathbf{A}(R)$, respectively $\mathbf{B}(R)$, as full subcategories. While $\mathbf{A}(R)$ enjoys a strong relation to Gorenstein projective modules, our main result is that $\mathbf{A}^s(R)$ has a similarly close relation to *homomorphisms* of Gorenstein projective modules.

This result is set in the wider context of a theory which shows that the two new categories inhabit a universe with strong symmetry properties.

Background on Auslander and Bass categories. Recall that the Auslander category $\mathbf{A}(R)$ can be characterized in terms of totally acyclic complexes of projective modules. Such a complex P consists of projective modules, is exact, and has the property that $\mathrm{Hom}_R(P, Q)$ is exact for each projective module Q . It was proved in [3, sec. 4] that a complex is in $\mathbf{A}(R)$ if and only if its homology is bounded and the left-tail of its projective resolution is equal to the left-tail of a totally acyclic complex of projective modules (all differentials point to the right).

The left-tails of totally acyclic complexes of projective modules are precisely the projective resolutions of so-called Gorenstein projective modules; this is immediate from the definition of a Gorenstein projective module as a cycle module of a totally acyclic complex of projectives, see [4]. This leads to the expectation that if we remove from $\mathbf{A}(R)$ a suitable “finite” part, leaving only the tails of projective resolutions, then we should get a category of Gorenstein projective modules.

Indeed, the homotopy category $\mathbf{K}^b(\mathrm{Prj} R)$ of bounded complexes of projective modules can be viewed as a subcategory of $\mathbf{A}(R)$, and we can remove it by forming the Verdier quotient $\mathbf{A}(R)/\mathbf{K}^b(\mathrm{Prj} R)$. On the other hand, the Gorenstein projective modules form a Frobenius category $\mathbf{GProj}(R)$, and there is a stable category $\underline{\mathbf{GProj}}(R)$ obtained by dividing out homomorphisms which factor through projective modules. It is not hard to show that there is an equivalence of triangulated

categories

$$\underline{\mathbf{GProj}}(R) \xrightarrow{\sim} \mathbf{A}(R)/\mathbf{K}^b(\mathbf{Prj} R). \quad (1)$$

Symmetric Auslander and Bass categories. The main result of this paper is a higher analogue of the above phenomenon. Let $\mathbf{K}(\mathbf{Prj} R)$ be the homotopy category of complexes of projective modules. We define the symmetric Auslander category $\mathbf{A}^s(R)$ to be the full subcategory of $\mathbf{K}(\mathbf{Prj} R)$ consisting of complexes whose left- and right-tails are equal to the left- and right-tails of totally acyclic complexes of projective modules.

Our main result is the following.

Theorem A. *There is an equivalence of triangulated categories*

$$\underline{\mathbf{GMor}}(R) \xrightarrow{\sim} \mathbf{A}^s(R)/\mathbf{K}^b(\mathbf{Prj} R).$$

Here $\underline{\mathbf{GMor}}(R)$ is the stable category of Gorenstein projective objects in $\mathbf{Mor}(R)$, the abelian category of homomorphisms of R -modules. Note that there is an equivalence of categories between $\mathbf{Mor}(R)$ and $\mathbf{Mod} T_2(R)^{\text{op}}$, the category of right-modules over the upper triangular matrix ring $T_2(R)$; cf. [1]. This implies that $\underline{\mathbf{GMor}}(R)$ is equivalent to the stable category of Gorenstein projective right-modules over $T_2(R)$.

On the other hand, we will show that the objects in $\underline{\mathbf{GMor}}(R)$ are precisely the injective homomorphisms between Gorenstein projective R -modules which have Gorenstein projective cokernels. Hence, whereas the Auslander category $\mathbf{A}(R)$ is related to Gorenstein projective modules via equation (1), the symmetric Auslander category $\mathbf{A}^s(R)$ is similarly related to *homomorphisms* of Gorenstein projective modules via Theorem A.

To prove the theorem, we develop a theory for the symmetric Auslander and Bass categories. One of the highlights is that $\mathbf{A}^s(R)$ is, indeed, a highly symmetric object. Namely, the quotient $\mathbf{A}^s(R)/\mathbf{K}^b(\mathbf{Prj} R)$ permits a so-called triangle of recollements $(\mathbf{U}, \mathbf{V}, \mathbf{W})$ as introduced in [6]. This means that $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are full subcategories of $\mathbf{A}^s(R)/\mathbf{K}^b(\mathbf{Prj} R)$, and that each of

$$(\mathbf{U}, \mathbf{V}), (\mathbf{V}, \mathbf{W}), (\mathbf{W}, \mathbf{U})$$

is a stable t-structure. It is not obvious, even in principle, that such a configuration is possible, but we show that

$$\begin{aligned} U &= A(R)/K^b(\text{Prj } R), \\ V &= K_{\text{tac}}(\text{Prj } R), \\ W &= S(B(R))/K^b(\text{Prj } R) \end{aligned} \tag{2}$$

work, where $K_{\text{tac}}(\text{Prj } R)$ is the full subcategory of $K(\text{Prj } R)$ consisting of totally acyclic complexes and S is a certain functor introduced in [7, sec. 4].

There are also several other results, among them the following.

Theorem B. *There are quasi-inverse equivalences of triangulated categories*

$$A^s(R) \xrightleftharpoons{\quad} B^s(R).$$

Let $K^{(b)}(\text{Prj } R)$ denote the full subcategory of $K(\text{Prj } R)$ consisting of complexes with bounded homology.

Theorem C. *There are inclusions*

$$A(R) \subseteq A^s(R) \subseteq K^{(b)}(\text{Prj } R).$$

The first inclusion is an equality if and only if each Gorenstein projective R -module is projective.

The second inclusion is an equality if and only if R is a Gorenstein ring.

Thus, the property that $A^s(R)$ is minimal, respectively maximal, characterises two interesting classes of rings.

Let us remark on two important sources of ideas for this paper. First, [6] originated the notion of a triangle of recollements and used it to get a version of Theorem A for finitely generated modules when R is a Gorenstein ring. The present paper can be viewed as extending these ideas. Secondly, while it is not obvious from the description above, we make extensive use of the machinery developed in [7] for homotopy categories of complexes of projective, respectively, injective modules and their relation to Auslander and Bass categories.

The paper is organised as follows: Section 1 briefly sketches the definitions and results we will use; most of them come from [7]. Section 2 proves Theorems B and C above (Theorems 2.7 and 2.9) and establishes the existence of the triangle of recollements described by equation

(2) (Theorem 2.10). Section 3 studies the category of homomorphisms $\text{Mor}(R)$ and its Gorenstein projective objects, and culminates in the proof of Theorem A (Theorem 3.12).

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1. BACKGROUND

This section recalls the tools we will use; most of them come from [7].

Setup 1.1. Throughout, R is a commutative noetherian ring with a dualizing complex D which is assumed to be a bounded complex of injective modules.

Dualizing complexes were introduced in [5], but see e.g. [3, sec. 1] for a contemporary introduction.

Remark 1.2. There are homotopy categories $\mathbf{K}(\text{Prj } R)$ and $\mathbf{K}(\text{Inj } R)$ of complexes of projective, respectively, injective modules. They have several important triangulated subcategories:

The subcategories of bounded complexes are denoted by $\mathbf{K}^b(\text{Prj } R)$ and $\mathbf{K}^b(\text{Inj } R)$. The subcategories of complexes with bounded homology are denoted by $\mathbf{K}^{(b)}(\text{Prj } R)$ and $\mathbf{K}^{(b)}(\text{Inj } R)$.

The subcategories of K-projective, respectively, K-injective complexes are denoted by $\mathbf{K}_{\text{prj}}(R)$ and $\mathbf{K}_{\text{inj}}(R)$; see [9].

The subcategories of totally acyclic complexes are denoted $\mathbf{K}_{\text{tac}}(\text{Prj } R)$ and $\mathbf{K}_{\text{tac}}(\text{Inj } R)$. Complexes X in $\mathbf{K}(\text{Prj } R)$ and Y in $\mathbf{K}(\text{Inj } R)$ are called totally acyclic if they are exact and $\text{Hom}_R(X, P)$ and $\text{Hom}_R(I, Y)$ are exact for each projective module P and each injective module I .

Remark 1.3. Consider the subcategories $\mathbf{K}_{\text{prj}}(R) \subseteq \mathbf{K}(\text{Prj } R)$ and $\mathbf{K}_{\text{inj}}(R) \subseteq \mathbf{K}(\text{Inj } R)$. By [7, sec. 7], the inclusion functors, which we will denote by inc , are parts of adjoint pairs of functors,

$$\mathbf{K}_{\text{prj}}(R) \begin{array}{c} \xrightarrow{\text{inc}} \\ \xleftarrow{p} \end{array} \mathbf{K}(\text{Prj } R) \quad \text{and} \quad \mathbf{K}_{\text{inj}}(R) \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{\text{inc}} \end{array} \mathbf{K}(\text{Inj } R).$$

In the terminology of [8, chp. 9], the existence of the right adjoint p places us in a situation of Bousfield localization, and accordingly, the counit morphism of the adjoint pair (inc, p) can be completed to a distinguished triangle

$$pX \xrightarrow{\epsilon_P} X \longrightarrow aX \longrightarrow$$

which depends functorially on X . Both p and a are triangulated functors. Dually, the unit morphism of the adjoint pair (i, inc) can be completed to a distinguished triangle

$$bY \longrightarrow Y \xrightarrow{\eta_Y} iY \longrightarrow$$

which depends functorially on Y .

Remark 1.4. By [7, thm. 4.2] there are quasi-inverse equivalences of categories

$$\mathbf{K}(\text{Prj } R) \xrightleftharpoons[S]{T} \mathbf{K}(\text{Inj } R)$$

where $T(-) = D \otimes_R -$ and $S = q \circ \text{Hom}_R(D, -)$. The functor q is right-adjoint to the inclusion $\mathbf{K}(\text{Prj } R) \rightarrow \mathbf{K}(\text{Flat } R)$ where $\mathbf{K}(\text{Flat } R)$ is the homotopy category of complexes of flat modules.

Remark 1.5. Let us recall the following from [2]. The derived category $\mathbf{D}(R)$ supports an adjoint pair of functors

$$\mathbf{D}(R) \xrightleftharpoons[\text{RHom}_R(D, -)]{D \overset{\text{L}}{\otimes}_R -} \mathbf{D}(R).$$

The Auslander category of R is the triangulated subcategory defined in terms of the unit η by

$$\mathbf{A}(R) = \left\{ X \in \mathbf{D}(R) \left| \begin{array}{l} X \text{ and } D \overset{\text{L}}{\otimes}_R X \text{ have bounded homology;} \\ X \xrightarrow{\eta_X} \text{RHom}_R(D, D \overset{\text{L}}{\otimes}_R X) \text{ is an isomorphism} \end{array} \right. \right\}$$

and the Bass category of R is the triangulated subcategory defined in terms of the counit ϵ by

$$\mathbf{B}(R) = \left\{ Y \in \mathbf{D}(R) \left| \begin{array}{l} Y \text{ and } \text{RHom}_R(D, Y) \text{ have bounded homology;} \\ D \overset{\text{L}}{\otimes}_R \text{RHom}_R(D, Y) \xrightarrow{\epsilon_Y} Y \text{ is an isomorphism} \end{array} \right. \right\}.$$

The functors $D \overset{\text{L}}{\otimes}_R -$ and $\text{RHom}_R(D, -)$ restrict to quasi-inverse equivalences between $\mathbf{A}(R)$ and $\mathbf{B}(R)$.

The canonical functors $\mathbf{K}_{\text{prj}}(R) \rightarrow \mathbf{D}(R)$ and $\mathbf{K}_{\text{inj}}(R) \rightarrow \mathbf{D}(R)$ are equivalences, and this permits us to view $\mathbf{A}(R)$ as a full subcategory

of $K_{\text{prj}}(R)$ and hence of $K(\text{Prj } R)$, and $B(R)$ as a full subcategory of $K_{\text{inj}}(R)$ and hence of $K(\text{Inj } R)$. As such, the adjoint functors

$$K_{\text{prj}}(R) \begin{array}{c} \xrightarrow{iT} \\ \xleftarrow{pS} \end{array} K_{\text{inj}}(R)$$

restrict to a pair of quasi-inverse equivalences between $A(R)$ and $B(R)$ by [7, prop. 7.2].

See [3, sec. 1] for an alternative review of Auslander and Bass categories.

Definition 1.6. Let \mathcal{T} be a triangulated category. A stable t-structure on \mathcal{T} is a pair of full subcategories $(\mathcal{U}, \mathcal{V})$ such that

- (i) $\Sigma \mathcal{U} = \mathcal{U}$, $\Sigma \mathcal{V} = \mathcal{V}$.
- (ii) $\text{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.
- (iii) For each T in \mathcal{T} there exist U in \mathcal{U} and V in \mathcal{V} and a distinguished triangle $U \rightarrow T \rightarrow V \rightarrow$.

A triangle of recollements in \mathcal{T} is a triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ such that each of $(\mathcal{U}, \mathcal{V})$, $(\mathcal{V}, \mathcal{W})$, $(\mathcal{W}, \mathcal{U})$ is a stable t-structure.

Let \mathcal{T}' be another triangulated category with a triangle of recollements $(\mathcal{U}', \mathcal{V}', \mathcal{W}')$ and let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be a triangulated functor. We say that F sends $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ to $(\mathcal{U}', \mathcal{V}', \mathcal{W}')$ if $F(\mathcal{U}) \subseteq \mathcal{U}'$, $F(\mathcal{V}) \subseteq \mathcal{V}'$, $F(\mathcal{W}) \subseteq \mathcal{W}'$.

2. SYMMETRIC AUSLANDER AND BASS CATEGORIES

This section develops a theory of symmetric Auslander and Bass categories. It proves Theorems B and C from the Introduction, and establishes the existence of the triangle of recollements described by equation (2) (Theorems 2.7, 2.9, and 2.10).

For the rest of the paper, an unadorned K stands for $K(\text{Prj } R)$. We combine this in an obvious way with various embellishments to form K^b , $K^{(b)}$, K_{prj} , and K_{tac} . Likewise, unadorned categories such as A , B , and D stand for $A(R)$, $B(R)$, and $D(R)$.

In the following definition, $X * Y$ denotes the full subcategory of objects C which sit in distinguished triangles $X \rightarrow C \rightarrow Y \rightarrow$ with X in \mathcal{X} and Y in \mathcal{Y} .

Definition 2.1. The *symmetric Auslander category* \mathbf{A}^s and the *symmetric Bass category* \mathbf{B}^s of R are the full subcategories of $\mathbf{K}(\text{Prj } R)$ and $\mathbf{K}(\text{Inj } R)$ defined by

$$\mathbf{A}^s = S(\mathbf{B}) * \mathbf{A} \text{ and } \mathbf{B}^s = \mathbf{B} * T(\mathbf{A})$$

where S and T are the functors from [7] described in Remark 1.4.

Remark 2.2. By [3, thm. 4.1], the subcategory \mathbf{A} of \mathbf{K} consists of complexes isomorphic to right-bounded complexes of projective modules whose left-tail is equal to the left-tail of a complete projective resolution.

Using the theory of [7], one can show that similarly, $S(\mathbf{B})$ consists of complexes isomorphic to left-bounded complexes of projective modules whose right-tail is equal to the right-tail of a complete projective resolution.

From this it follows that \mathbf{A}^s consists of complexes isomorphic to complexes of projective modules both of whose tails are equal to the tails of complete projective resolutions.

Similar remarks apply to \mathbf{B}^s , and this is one of the reasons for the terminology “symmetric Auslander and Bass categories”.

Remark 2.3. The following lemma and most of the other results in this section will only be given for \mathbf{A}^s , but there are dual versions for \mathbf{B}^s with similar proofs.

Lemma 2.4. *Let C be in \mathbf{K} . Then C is in \mathbf{A}^s if and only if the following conditions are satisfied.*

- (i) C and TC have bounded homology.
- (ii) The mapping cone of $pC \xrightarrow{\epsilon_C} C$ is totally acyclic.
- (iii) The mapping cone of $TC \xrightarrow{\eta_{TC}} iTC$ is totally acyclic.

Proof. “Only if”: Suppose that C is in \mathbf{A}^s . By definition, there is a distinguished triangle

$$SB \rightarrow C \rightarrow A \rightarrow$$

in \mathbf{K} with B in \mathbf{B} and A in \mathbf{A} . All of SB , A , $TSB \cong B$, and TA have bounded homology, so the same is true for C and TC , proving condition (i).

By Remark 1.3, the distinguished triangle induces the following commutative diagram where each row and each column is a distinguished

triangle.

$$\begin{array}{ccccccc}
 pSB & \longrightarrow & pC & \longrightarrow & pA & \longrightarrow & \\
 \epsilon_{SB} \downarrow & & \epsilon_C \downarrow & & \epsilon_A \downarrow & & \\
 SB & \longrightarrow & C & \longrightarrow & A & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 aSB & \xrightarrow{\alpha} & aC & \longrightarrow & aA & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & &
 \end{array}$$

Since A is K-projective, ϵ_A is an isomorphism. Hence aA is zero so α is an isomorphism. But B is in \mathbf{B} so aSB is totally acyclic by [7, prop. 7.4], and so aC is totally acyclic, proving condition (ii). A similar argument proves condition (iii).

“If”: Suppose that conditions (i) through (iii) hold. Hard truncation gives a distinguished triangle

$$C^{\geq 0} \rightarrow C \rightarrow C^{< 0} \rightarrow$$

in \mathbf{K} . We aim to show that $C^{\geq 0}$ is in $S(\mathbf{B})$ and that $C^{< 0}$ is in \mathbf{A} whence C is in \mathbf{A}^s .

Set

$$B = T(C^{\geq 0}) = D \otimes_R C^{\geq 0}$$

so $SB = ST(C^{\geq 0}) \cong C^{\geq 0}$. Since $C^{\geq 0}$ is a left-bounded complex of projective modules and D is a bounded complex of injective modules, B is a left-bounded complex of injective modules. In particular, it is K-injective.

Since D is bounded, the complexes B and $TC = D \otimes_R C$ agree in high cohomological degrees. But B is left-bounded and TC has bounded homology by condition (i), so it follows that B has bounded homology. Also, B is K-injective so $\mathrm{RHom}_R(D, B)$ can be computed as $\mathrm{Hom}_R(D, B)$, but

$$\mathrm{Hom}_R(D, B) \stackrel{(a)}{\simeq} q \circ \mathrm{Hom}_R(D, B) = SB \cong C^{\geq 0}$$

where the quasi-isomorphism (a) is by [7, thm. 2.7]. Since the homology of $C^{\geq 0}$ is bounded, so is the homology of $\mathrm{RHom}_R(D, B)$.

As above, the distinguished triangle induces the following commutative diagram where each row and each column is a distinguished triangle.

$$\begin{array}{ccccccc}
 pC^{\geq 0} & \longrightarrow & pC & \longrightarrow & pC^{< 0} & \longrightarrow & \\
 \epsilon_{C^{\geq 0}} \downarrow & & \epsilon_C \downarrow & & \epsilon_{C^{< 0}} \downarrow & & \\
 C^{\geq 0} & \longrightarrow & C & \longrightarrow & C^{< 0} & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 aC^{\geq 0} & \xrightarrow{\beta} & aC & \longrightarrow & aC^{< 0} & \longrightarrow & \\
 \downarrow & & \downarrow & & \downarrow & &
 \end{array}$$

Since $C^{< 0}$ is a right-bounded complex of projective modules it is K-projective and so $\epsilon_{C^{< 0}}$ is an isomorphism. Hence $aC^{< 0}$ is zero so β is an isomorphism. But aC is totally acyclic by condition (ii), and so $aC^{\geq 0}$ is totally acyclic. Since $SB \cong C^{\geq 0}$, it follows from [7, prop. 7.4] that B is in \mathbf{B} and so $C^{\geq 0}$ is in $S(\mathbf{B})$.

A similar argument proves that $C^{< 0}$ is in \mathbf{A} . \square

Proposition 2.5. *The category \mathbf{A}^s is a triangulated subcategory of \mathbf{K} , and there are inclusions of triangulated subcategories*

$$\mathbf{K}_{\text{tac}} \subseteq \mathbf{A}^s \subseteq \mathbf{K}^{(b)}.$$

Proof. It is well known that \mathbf{K}_{tac} and $\mathbf{K}^{(b)}$ are triangulated subcategories of \mathbf{K} .

Conditions (i) through (iii) of Lemma 2.4 respect mapping cones, so \mathbf{A}^s is a triangulated subcategory of \mathbf{K} .

The second inclusion of the proposition is immediate from Lemma 2.4(i), and the first one follows from Lemma 2.4(i)–(iii) combined with the fact that T sends totally acyclic complexes to totally acyclic complexes by [7, prop. 5.9(1)]. \square

Remark 2.6. We owe the following observations based on Lemma 2.4 to Srikanth Iyengar.

The Auslander and Bass categories \mathbf{A} and \mathbf{B} also exist in versions $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ without boundedness conditions [7, 7.1]. With small modifications, the proof of Lemma 2.4 shows that membership of $S(\widehat{\mathbf{B}}) * \widehat{\mathbf{A}}$ is characterised by conditions (ii) and (iii) of the Lemma.

It is immediate from Lemma 2.4 that $A * S(B)$ is contained in $A^s = S(B) * A$. This is a bit surprising since one would not normally expect any inclusion between categories of the form $X * Y$ and $Y * X$.

We do not know if $A * S(B)$ is triangulated, but it will often be considerably smaller than $S(B) * A$ since K_{tac} is contained in $S(B) * A$ by Proposition 2.5 while it is easy to show that the intersection of $A * S(B)$ with K_{tac} is zero.

Theorem 2.7. *The functors T and S restrict to quasi-inverse equivalences of triangulated categories*

$$A^s \begin{array}{c} \xrightarrow{T} \\ \xleftarrow{S} \end{array} B^s.$$

Proof. This is immediate from the definition of A^s and B^s because T and S are quasi-inverse equivalences of triangulated categories. \square

Theorem 2.8. (i) *The category A^s has stable t -structures*

$$(A, K_{\text{tac}}(\text{Prj } R)) \text{ and } (K_{\text{tac}}(\text{Prj } R), S(B)).$$

(ii) *The category B^s has stable t -structures*

$$(K_{\text{tac}}(\text{Inj } R), B) \text{ and } (T(A), K_{\text{tac}}(\text{Inj } R)).$$

Proof. The first of the stable t -structures in part (i) can be established as follows.

The category A^s contains A by definition and K_{tac} by Proposition 2.5. Each A in A is K -projective, so a morphism $A \rightarrow U$ with U in K_{tac} is zero.

Existence of the first stable t -structure will thus follow if we can prove $A^s = A * K_{\text{tac}}$.

For C in A^s , there is a distinguished triangle $SB \rightarrow C \rightarrow A \rightarrow$ with B in B and A in A . Turning the triangle gives a distinguished triangle $\Sigma^{-1}A \xrightarrow{\alpha} SB \rightarrow C \rightarrow A$.

There is also a distinguished triangle $pSB \xrightarrow{\epsilon_{SB}} SB \rightarrow U \rightarrow$ and U is totally acyclic by [7, prop. 7.4]. Since $\Sigma^{-1}A$ is in A , each morphism $\Sigma^{-1}A \rightarrow U$ is zero, and hence α lifts through ϵ_{SB} .

By the octahedral axiom, there is hence a commutative diagram in which each row and each column is a distinguished triangle,

$$\begin{array}{ccccccc}
 \Sigma^{-1}pSB & \longrightarrow & \Sigma^{-1}SB & \longrightarrow & \Sigma^{-1}U & \longrightarrow & pSB \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma^{-1}A' & \longrightarrow & 0 & \longrightarrow & A' & \xlongequal{\quad} & A' \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma^{-1}A & \xrightarrow{\alpha} & SB & \longrightarrow & C & \longrightarrow & A \\
 \downarrow & & \parallel & & \downarrow & & \downarrow \\
 pSB & \xrightarrow{\epsilon_{SB}} & SB & \longrightarrow & U & \longrightarrow & \Sigma pSB.
 \end{array}$$

Since B is in \mathbf{B} , the object pSB is in \mathbf{A} by [7, prop. 7.2]; see Remark 1.5. Since A is also in \mathbf{A} , it follows that A' is in \mathbf{A} . So the third column of the above diagram shows $\mathbf{A}^s = \mathbf{A} * \mathbf{K}_{\text{tac}}$, proving existence of the first stable t-structure in the theorem.

The first of the stable t-structures in part (ii) follows by an analogous argument using [7, prop. 7.3] instead of [7, prop. 7.2].

The second stable t-structure in part (i) is obtained by applying S to the first stable t-structure in part (ii). The second stable t-structure in part (ii) is obtained by applying T to the first stable t-structure in part (i). \square

Theorem 2.9. *There are inclusions*

$$\mathbf{A} \subseteq \mathbf{A}^s \subseteq \mathbf{K}^{(\text{b})}.$$

The first inclusion is an equality if and only if each Gorenstein projective R -module is projective.

The second inclusion is an equality if and only if R is a Gorenstein ring.

Proof. The first inclusion is clear from the definition of \mathbf{A}^s , and the second holds by Proposition 2.5.

The claim on the first inclusion: The first stable t-structure of Theorem 2.8 shows that $\mathbf{A}^s = \mathbf{A}$ is equivalent to $\mathbf{K}_{\text{tac}} = 0$. This happens if and only if each totally acyclic complex is split exact, that is, if and only if each Gorenstein projective module is projective.

The claim on the second inclusion: First, suppose that $\mathbf{A}^s = \mathbf{K}^{(\text{b})}$. Let M be an R -module with projective resolution C ; it follows that C is

in \mathbf{A}^s . Consider the distinguished triangle $A \rightarrow C \rightarrow U \rightarrow$ with A in \mathbf{A} and U in \mathbf{K}_{tac} which exists by Theorem 2.8. Since U is exact, the homology of A is M so the K -projective complex A is a projective resolution of M . This shows that for each module M , the projective resolution is in \mathbf{A} , hence the Gorenstein projective dimension of M is finite by [3, thm. 4.1], and hence R is Gorenstein.

Secondly, suppose that R is Gorenstein and let C be in $\mathbf{K}^{(b)}$. We will show that C is in \mathbf{A}^s by showing that C satisfies the three conditions of Lemma 2.4.

In condition (i), by definition, C has bounded homology. Since R is Gorenstein, D can be taken to be an injective resolution of R . Hence there is a quasi-isomorphism $R \rightarrow D$ of bounded complexes, and since C consists of projective modules, it follows that there is a quasi-isomorphism $R \otimes_R C \rightarrow D \otimes_R C$. So $TC = D \otimes_R C$ also has bounded homology.

Conditions (ii) and (iii) hold because the relevant mapping cones are acyclic, and over a Gorenstein ring this implies that they are totally acyclic; see [7, cor. (5.5)]. \square

In the following theorem, note that \mathbf{K}_{tac} is a triangulated subcategory of \mathbf{A}^s which can also be viewed as a triangulated subcategory of the Verdier quotient $\mathbf{A}^s/\mathbf{K}^b$ since there are only zero morphisms from \mathbf{K}^b to \mathbf{K}_{tac} .

Theorem 2.10. *The category $\mathbf{A}^s/\mathbf{K}^b$ has a triangle of recollements*

$$(\mathbf{A}/\mathbf{K}^b, \mathbf{K}_{\text{tac}}, S(\mathbf{B})/\mathbf{K}^b).$$

That is, it has stable t-structures

$$(\mathbf{A}/\mathbf{K}^b, \mathbf{K}_{\text{tac}}), (\mathbf{K}_{\text{tac}}, S(\mathbf{B})/\mathbf{K}^b), (S(\mathbf{B})/\mathbf{K}^b, \mathbf{A}/\mathbf{K}^b).$$

Proof. The first two stable t-structures follow from the stable t-structures of Theorem 2.8 by [6].

Let us show that the third structure exists. By definition, $\mathbf{A}^s = S(\mathbf{B}) * \mathbf{A}$, and this implies $\mathbf{A}^s/\mathbf{K}^b = (S(\mathbf{B})/\mathbf{K}^b) * (\mathbf{A}/\mathbf{K}^b)$.

It is therefore enough to show that each morphism $S(B) \rightarrow A$ in $\mathbf{K}^{(b)}/\mathbf{K}^b$ with $S(B)$ in $S(\mathbf{B})/\mathbf{K}^b$ and A in \mathbf{A}/\mathbf{K}^b must be zero. Such a morphism is represented by a diagram $S(B) \rightarrow A' \leftarrow A$ in $\mathbf{K}^{(b)}$ where the mapping cone of $A \rightarrow A'$ is in \mathbf{K}^b . In particular, the mapping cone is in \mathbf{A} , so A' is also in \mathbf{A} whence A' is isomorphic to a right-bounded complex of projective modules. However, $S(B)$ is isomorphic

to a left-bounded complex of projective modules, and it easily follows that the morphism $S(B) \rightarrow A'$ factors through an object of \mathbf{K}^b . Hence this morphism becomes zero in $\mathbf{K}^{(b)}/\mathbf{K}^b$, and so the original morphism $S(B) \rightarrow A$ in $\mathbf{K}^{(b)}/\mathbf{K}^b$ is zero as desired. \square

3. THE CATEGORY OF HOMOMORPHISMS

This section proves our main result, Theorem A from the Introduction (Theorem 3.12).

Definition 3.1. We let \mathbf{Mor} denote the category of homomorphisms of R -modules.

The objects of \mathbf{Mor} are the homomorphisms of R -modules.

The morphisms of \mathbf{Mor} are defined as follows: A morphism f from $X_\alpha \xrightarrow{\alpha} T_\alpha$ to $X_\beta \xrightarrow{\beta} T_\beta$ is a pair (f_X, f_T) of homomorphisms of R -modules $X_\alpha \xrightarrow{f_X} X_\beta$ and $T_\alpha \xrightarrow{f_T} T_\beta$ such that there is a commutative square

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_X} & X_\beta \\ \alpha \downarrow & & \downarrow \beta \\ T_\alpha & \xrightarrow{f_T} & T_\beta. \end{array}$$

Remark 3.2. Given an object $X_\alpha \xrightarrow{\alpha} T_\alpha$ in \mathbf{Mor} , we will denote the cokernel of α by N_α .

Observe that a morphism f in \mathbf{Mor} induces a commutative diagram of R -modules with exact rows,

$$\begin{array}{ccccccc} X_\alpha & \xrightarrow{\alpha} & T_\alpha & \longrightarrow & N_\alpha & \longrightarrow & 0 \\ f_X \downarrow & & f_T \downarrow & & \downarrow f_N & & \\ X_\beta & \xrightarrow{\beta} & T_\beta & \longrightarrow & N_\beta & \longrightarrow & 0. \end{array}$$

Remark 3.3. A complex $\pi = \cdots \rightarrow \pi^i \xrightarrow{d_\pi^i} \pi^{i+1} \cdots$ in \mathbf{Mor} implies a chain map π between complexes of R -modules,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X_{\pi^i} & \longrightarrow & X_{\pi^{i+1}} & \longrightarrow & \cdots \\ & & \pi^i \downarrow & & \pi^{i+1} \downarrow & & \\ \cdots & \longrightarrow & T_{\pi^i} & \longrightarrow & T_{\pi^{i+1}} & \longrightarrow & \cdots. \end{array}$$

It is not hard to check that the projective objects of \mathbf{Mor} are precisely the split injections between projective R -modules. Hence, if π is a complex of projective objects in \mathbf{Mor} , then there is an exact sequence

$$0 \rightarrow X_\pi \xrightarrow{\pi} T_\pi \rightarrow N_\pi \rightarrow 0 \quad (1)$$

of complexes of projective R -modules.

The proof of the following lemma is straightforward.

Lemma 3.4. *Let $\alpha \xrightarrow{f} \beta$ be a morphism in the category \mathbf{Mor} . Let M be an R -module and consider the zero homomorphism $0 \xrightarrow{0^M} M$ and the identity $M \xrightarrow{1_M} M$ as objects of \mathbf{Mor} . Then we have the following.*

(i) *There are vertical isomorphisms giving a commutative square*

$$\begin{array}{ccc} \mathrm{Hom}_R(N_\beta, M) & \xrightarrow{\mathrm{Hom}_R(f_N, M)} & \mathrm{Hom}_R(N_\alpha, M) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Hom}_{\mathbf{Mor}}(\beta, 0^M) & \xrightarrow{\mathrm{Hom}_{\mathbf{Mor}}(f, 0^M)} & \mathrm{Hom}_{\mathbf{Mor}}(\alpha, 0^M). \end{array}$$

(ii) *There are vertical isomorphisms giving a commutative square*

$$\begin{array}{ccc} \mathrm{Hom}_R(T_\beta, M) & \xrightarrow{\mathrm{Hom}_R(f_T, M)} & \mathrm{Hom}_R(T_\alpha, M) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Hom}_{\mathbf{Mor}}(\beta, 1_M) & \xrightarrow{\mathrm{Hom}_{\mathbf{Mor}}(f, 1_M)} & \mathrm{Hom}_{\mathbf{Mor}}(\alpha, 1_M). \end{array}$$

Lemma 3.5. *A complex π of projective objects in \mathbf{Mor} is totally acyclic if and only if each of the complexes*

$$\begin{aligned} X_\pi &= \cdots \longrightarrow X_{\pi^i} \longrightarrow X_{\pi^{i+1}} \longrightarrow \cdots, \\ T_\pi &= \cdots \longrightarrow T_{\pi^i} \longrightarrow T_{\pi^{i+1}} \longrightarrow \cdots \end{aligned}$$

belongs to $\mathbf{K}_{\mathrm{tac}}$.

Proof. Let φ be a projective object of \mathbf{Mor} . Remark 3.3 says that φ is a split injection of projective R -modules, so there are projective R -modules P and P' such that $\varphi = 0^P \oplus 1_{P'}$. The complex $\mathrm{Hom}_{\mathbf{Mor}}(\pi, \varphi)$ is acyclic if and only if both $\mathrm{Hom}_{\mathbf{Mor}}(\pi, 0^P)$ and $\mathrm{Hom}_{\mathbf{Mor}}(\pi, 1_{P'})$ are acyclic. By Lemma 3.4, this is equivalent to having both complexes $\mathrm{Hom}_R(T_\pi, P)$ and $\mathrm{Hom}_R(N_\pi, P')$ acyclic.

Therefore π is totally acyclic if and only if T_π and N_π are both totally acyclic, which by the sequence (1) is equivalent to both of T_π and X_π being totally acyclic. \square

Corollary 3.6. *The Gorenstein projective objects of \mathbf{Mor} are the injective homomorphisms between Gorenstein projective R -modules which have Gorenstein projective cokernels.*

Proof. A Gorenstein projective object in \mathbf{Mor} is a cycle of a totally acyclic complex of projective objects of \mathbf{Mor} . It follows easily from Lemma 3.5 that it is an injective homomorphism between Gorenstein projective R -modules, and that the cokernel is Gorenstein projective.

Conversely, let X_α and T_α be Gorenstein projective R -modules and suppose that $X_\alpha \xrightarrow{\alpha} T_\alpha$ is an injective homomorphism with Gorenstein projective cokernel. Using the Horseshoe Lemma, the short exact sequence $0 \rightarrow X_\alpha \xrightarrow{\alpha} T_\alpha \rightarrow N_\alpha \rightarrow 0$ gives a short exact sequence of complete projective resolutions

$$0 \longrightarrow P_{X_\alpha} \xrightarrow{\pi_\alpha} P_{T_\alpha} \longrightarrow P_{N_\alpha} \longrightarrow 0.$$

Lemma 3.5 says that $P_{X_\alpha} \xrightarrow{\pi_\alpha} P_{T_\alpha}$ can be viewed as a totally acyclic complex of projective objects of \mathbf{Mor} , and it is clear that it is a complete projective resolution of $X_\alpha \xrightarrow{\alpha} T_\alpha$ which is hence a Gorenstein projective object of \mathbf{Mor} . \square

Definition 3.7. We denote the full subcategory of Gorenstein projective objects in \mathbf{Mor} by \mathbf{GMor} . Inside \mathbf{GMor} , we consider the following full subcategories \mathbf{GMor}^p , \mathbf{GMor}^0 , and \mathbf{GMor}^1 .

- (i) \mathbf{GMor}^p consists of injective homomorphisms $X \xrightarrow{\iota_X} P$ where X is Gorenstein projective and P is projective.
- (ii) \mathbf{GMor}^0 consists of zero homomorphisms $0 \xrightarrow{0^T} T$ where T is Gorenstein projective.
- (iii) \mathbf{GMor}^1 consists of identity homomorphisms $X \xrightarrow{1_M} X$ where X is Gorenstein projective.

There are corresponding stable categories which are defined by dividing out the morphisms which factor through a projective object. The stable categories are denoted by underlining. The category $\underline{\mathbf{GMor}}$ is triangulated, and $\underline{\mathbf{GMor}}^p$, $\underline{\mathbf{GMor}}^0$, and $\underline{\mathbf{GMor}}^1$ are triangulated subcategories.

Theorem 3.8. *The category $\underline{\mathbf{GMor}}$ has a triangle of recollements*

$$(\underline{\mathbf{GMor}}^p, \underline{\mathbf{GMor}}^1, \underline{\mathbf{GMor}}^0).$$

That is, it has stable t -structures

$$(\underline{\mathbf{GMor}}^p, \underline{\mathbf{GMor}}^1), (\underline{\mathbf{GMor}}^1, \underline{\mathbf{GMor}}^0), (\underline{\mathbf{GMor}}^0, \underline{\mathbf{GMor}}^p).$$

Proof. It is enough to show that each of the following categories: (i) $\underline{\mathbf{GMor}}^p * \underline{\mathbf{GMor}}^1$, (ii) $\underline{\mathbf{GMor}}^1 * \underline{\mathbf{GMor}}^0$, and (iii) $\underline{\mathbf{GMor}}^0 * \underline{\mathbf{GMor}}^p$ is equal to $\underline{\mathbf{GMor}}$.

Let $X_\alpha \xrightarrow{\alpha} T_\alpha$ be an object of $\underline{\mathbf{GMor}}$ and consider the exact sequence $0 \rightarrow X_\alpha \xrightarrow{\alpha} T_\alpha \xrightarrow{\beta} N_\alpha \rightarrow 0$ of G-projective R -modules. There exist injective homomorphisms $\iota_{T_\alpha} : T_\alpha \rightarrow P$ and $\iota_{N_\alpha} : N_\alpha \rightarrow P'$ with projective R -modules P and P' .

(i) The commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_\alpha & \xrightarrow{\alpha} & T_\alpha & \xrightarrow{\beta} & N_\alpha \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \iota_{N_\alpha} \\ 0 & \longrightarrow & T_\alpha & \xrightarrow{\begin{pmatrix} 1 \\ -\iota_{N_\alpha} \beta \end{pmatrix}} & T_\alpha \oplus P' & \xrightarrow{(\iota_{N_\alpha} \beta, 1)} & P' \longrightarrow 0 \end{array}$$

induces a distinguished triangle in $\underline{\mathbf{GMor}}$

$$\Sigma^{-1} \underline{\iota_{N_\alpha}} \rightarrow \underline{\alpha} \rightarrow \underline{1_{T_\alpha}} \rightarrow$$

with $\Sigma^{-1} \underline{\iota_{N_\alpha}}$ in $\underline{\mathbf{GMor}}^p$ and $\underline{1_{T_\alpha}}$ in $\underline{\mathbf{GMor}}^1$.

(ii) The commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_\alpha & \xlongequal{\quad} & X_\alpha & \longrightarrow & 0 \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha & & \downarrow 0^{N_\alpha} \\ 0 & \longrightarrow & X_\alpha & \xrightarrow{\alpha} & T_\alpha & \xrightarrow{\beta} & N_\alpha \longrightarrow 0 \end{array}$$

induces a distinguished triangle in $\underline{\mathbf{GMor}}$

$$\underline{1_{X_\alpha}} \rightarrow \underline{\alpha} \rightarrow \underline{0^{N_\alpha}} \rightarrow$$

with $\underline{1_{X_\alpha}}$ in $\underline{\mathbf{GMor}}^1$ and $\underline{0^{N_\alpha}}$ in $\underline{\mathbf{GMor}}^0$.

(iii) The commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X_\alpha & \xlongequal{\quad} & X_\alpha & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \iota_{T_\alpha} \alpha & & \downarrow 0^{\Sigma T_\alpha} \\
 0 & \longrightarrow & T_\alpha & \xrightarrow{\quad \iota_{T_\alpha} \quad} & P & \longrightarrow & \Sigma T_\alpha \longrightarrow 0
 \end{array}$$

induces a distinguished triangle in $\underline{\mathbf{GMor}}$

$$\underline{0^{T_\alpha}} \rightarrow \underline{\alpha} \rightarrow \underline{\iota_{T_\alpha} \alpha} \rightarrow$$

with $\underline{0^{T_\alpha}}$ in $\underline{\mathbf{GMor}}^0$ and $\underline{\iota_{T_\alpha} \alpha}$ in $\underline{\mathbf{GMor}}^p$. \square

Let $X_\alpha \xrightarrow{\alpha} T_\alpha$ be an object of \mathbf{GMor} and consider complete projective resolutions P of X_α and \tilde{P} of T_α . In particular, there is a surjection $P^0 \xrightarrow{\rho} X_\alpha$ and an injection $T_\alpha \xrightarrow{\iota} \tilde{P}^1$. Let P_α denote the complex

$$\dots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{\iota \alpha \rho} \tilde{P}^1 \longrightarrow \tilde{P}^2 \longrightarrow \dots$$

Proposition 3.9 ([6, lemmas 4.2 and 4.3 and prop. 4.4]). *The operation $\alpha \mapsto P_\alpha$ gives a functor $\mathbf{GMor} \rightarrow \mathbf{A}^s$ which induces a triangulated functor*

$$\underline{P} : \underline{\mathbf{GMor}} \rightarrow \mathbf{A}^s / \mathbf{K}^b.$$

Lemma 3.10 ([6, lemmas 4.6 and 4.7]). (i) \underline{P} sends the triangle of recollements

$$(\underline{\mathbf{GMor}}^p, \underline{\mathbf{GMor}}^1, \underline{\mathbf{GMor}}^0)$$

to the triangle of recollements

$$(\mathbf{A} / \mathbf{K}^b, \mathbf{K}_{\text{tac}}, S(\mathbf{B}) / \mathbf{K}^b).$$

(ii) *The restriction of \underline{P} to $\underline{\mathbf{GMor}}^1$ is an equivalence of triangulated categories $\underline{\mathbf{GMor}}^1 \rightarrow \mathbf{K}_{\text{tac}}$.*

Proposition 3.11 ([6, Prop. 1.18]). *Let $(\mathbf{U}, \mathbf{V}, \mathbf{W})$ and $(\mathbf{U}', \mathbf{V}', \mathbf{W}')$ be triangles of recollements in \mathbf{T} and \mathbf{T}' respectively. Suppose the triangulated functor $F : \mathbf{T} \rightarrow \mathbf{T}'$ sends $(\mathbf{U}, \mathbf{V}, \mathbf{W})$ to $(\mathbf{U}', \mathbf{V}', \mathbf{W}')$. If the restriction $F|_{\mathbf{U}}$ is an equivalence of triangulated categories, then so is F .*

The following main theorem follows immediately by combining Lemma 3.10 and Proposition 3.11; compare with [6, lem. 4.7 and thm. 4.8].

Theorem 3.12. *The functor \underline{P} is an equivalence of triangulated categories*

$$\underline{\mathbf{GMor}} \rightarrow \mathbf{A}^s / \mathbf{K}^b.$$

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